5 September 2007 Note: The notation after Proposition 6.4 is still under reconstruction!

Imperfect Signaling
and the Local Credibility Test

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Abstract

In this paper we study equilibrium refinement in signaling models. By allowing for deviations by a pool of "nearby" types, we propose a Local Credibility Test (LCT) which gives consistent solutions for any positive, though not necessarily perfect correlation between the signal sender's true types (e.g., signaling cost) and the value to the signal receiver (e.g., marginal product). We identify conditions for an equilibrium to satisfy the LCT in both the finite and continuous type cases, and demonstrate that the conditions are identical as we take the limit in the finite type case. Intuitively, the conditions for an equilibrium to survive our LCT test require that a measure of signaling "effectiveness" is sufficiently high for every type and that the type distribution is not tilted upwards too much. We then apply the characterization results to several signaling applications.

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1. Introduction

Since the seminal work of Cho and Kreps (1987), various refinement concepts have been proposed to rank different equilibria in signaling games in terms of their "reasonableness". However, the mission is still far from being completed. In many applications, signals are "imperfect" in the sense that there is a positive yet imperfect correlation between the signal sender's true type (e.g., signaling cost) and the signal receiver's expected value (which then determines her response), see Riley (2001, 2002).[1] Consider a situation in which two of the sender types have a same signaling cost but quite different values to the receiver. If these two types do not observe their values to the receiver, they are effectively the same type, so the existing refinement concepts, such as the Cho and Kreps Intuitive Criterion, apply in the usual way. However, if these two types do observe their different values to the receiver, then the Intuitive Criterion is unable to rank equilibria. The reason is that if one of the two types likes a deviation, the other also likes it, hence no deviation is credible by a single type. This is highly unsatisfactory because the two cases are observationally equivalent.

The reason for the inconsistent solutions in the above example is that the existing refinement concepts focus on deviations by a single type only and do not consider deviations by a pool of types. Grossman and Perry (1986a,b), in a bargaining context, propose an equilibrium refinement concept strengthening the Cho-Kreps Intuitive criterion to allow pooling deviations. In this paper, in a general signaling model, we weaken the Grossman-Perry Criterion, and propose a "Local Credibility Test" (LCT) in which a possible deviation is interpreted as coming from those types whose equilibrium actions are nearby. We consider only local pooling deviations, first because they seem to

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[1] Imperfect correlation between the signaling cost type and the receiver's expected value naturally arises when the sender's "physical" characteristics are multidimensional. The sender and the receiver may have similar preferences over the characteristics, but place different weights on its different dimensions. In the Spence education signaling model, a worker's characteristics may consist of her analytical skills and social skills. Both skills can be important to her education cost as well as to her marginal product in the workplace, but their relative importance in the two clearly can differ. In the reserve price signaling model of Cai, Riley and Ye (2004), characteristics of an artwork in an auction include its quality, rarity, history, etc. The seller (who has private information about its characteristics and signals the information with reserve prices) may be mostly concerned with characteristics related to the artwork's secondary market value, while potential buyers (who buy for self consumption) may care more about its impact in the setting for which it is intended.

us the most natural, second because they have much of the power of global pooling deviations, and third because they are more easily analyzed.

Consider equilibrium of a signaling game. Suppose an out-of-equilibrium signal is observed. By the Cho-Kreps Intuitive Criterion, if one sender type can be strictly better off deviating to this signal from his equilibrium signal but all other types cannot, then such a deviation is credible for this type. The equilibrium is said to fail the Intuitive Criterion if there exists such a credible deviation. In addition to the requirement of the Intuitive Criterion, the Local Credibility Test allows the possibility of small pooling deviations. Specifically, imagine that those types whose equilibrium signals are nearby the observed out-of-equilibrium signal deviate to this signal from their equilibrium signals but all other types do not, and the receiver correctly "anticipates" such a pooling deviation and holds the right perception about the expected type of the pool. If under the receiver's right perception, all the nearby types can be strictly better off from the deviation but all other types cannot, then such a pooling deviation is credible and we say that the equilibrium fails the LCT . By allowing pooling deviations, the LCT can be easily applied to situations with imperfect correlation between the signaling cost type and the receiver's expected value.

More importantly, the Local Credibility Test does not always rule out pooling equilibria in favor of separating equilibria. We will argue that in some situations separating equilibria seem unreasonable while pooling equilibria can be rather appealing. Precisely in such situations, the LCT avoids selecting the unreasonable separating equilibria. Thus, unlike the existing refinement concepts that always rank separating equilibria above pooling equilibria, the LCT selects separating equilibria only when they are reasonable.

In the local credibility test, if a subset of types choose the same equilibrium signal, we consider deviations by all those types. As we shall see, there is typically a family of equilibria that satisfies the LCT. We also consider a strong LCT in which we consider deviations by all subsets of local types. This is very much in the spirit of the Grossman-Perry Sequential Perfect Nash Equilibrium, except that we consider only local deviations. We show that if there is an equilibrium satisfying the strong LCT, there can be no pools.

Consider a simple two type education-signaling model, in which the high type must take a quite costly signal (e.g., many years of unproductive education) to separate from the low type. Now suppose there is only one low type agent in every 5 million high type agents. In such a situation separation seems highly unreasonable, because without taking the costly signaling action an agent should not be perceived much differently from being the high type. By the LCT, it is easy to show that in any separating equilibrium a pooling deviation to some sufficiently low cost level of the signal is profitable to both types, so no separating equilibrium satisfies the LCT.

Many signaling applications are formulated in models with continuous types. Another advantage of the LCT is that it can be applied to both finite and continuous type cases equally well. We begin by formulating the concept of the LCT for the finite type models first, since the intuition is easier to present. Then we consider a discretization of the continuous type model, and take the limit as the discretization becomes finer. Later we study a family of continuous type models of which many commonly studied signaling applications such as the Spence education signaling model are members. We demonstrate that the conditions for an equilibrium to satisfy the LCT are identical in these two cases.

Another innovation of our analysis is to consider explicitly the sender's decision to participate in signaling. Economically this is important because potential entrants can influence signaling behavior of active senders in real world applications. Analytically, the existence of potential entrants helps ensure that senders of types slightly above the minimum signaling type do not want to deviate collectively to the minimum signal. We show that the only candidate equilibrium that can survive the strong LCT is a separating equilibrium that has the "right" minimum signaling type and the associated minimum signal. We then characterize conditions under which this equilibrium satisfies the strong LCT. The required conditions are intuitive. As long as a measure of signaling "effectiveness" is sufficiently high for every type above the minimum signaling type and the type distribution is not tilted upwards too much, the candidate equilibrium can survive our strong LCT test.

In the continuous type case, the set of equilibrium signals is dense so that out-of-equilibrium signals can be only found outside the set of equilibrium signals. However, thinking of the continuous type case as the limiting case of the finite type case with many close types, it is natural to generalize the concept of the LCT to the continuous type case. An equilibrium survives the LCT if no change in perception is credible in the following sense: for any possible signal (on- or off-equilibrium), if the revised perception is that the signal is from types of a small neighborhood of the immediate equilibrium type, it is profitable for the types in this neighborhood to deviate to this signal, but unprofitable for types out of this neighborhood to do so. Another way of thinking about this credibility test in the continuous type case is the following. If, for an on-equilibrium signal, there is such a deviation-perception pair, then those nearby types can credibly deviate to the particular on-equilibrium signal by throwing away ε amount of money. Since no other types would be willing to do so, this could convince the receiver that the deviating sender is indeed one of those nearby types, thus making the deviation-perception credible.

We derive conditions under which the LCT is satisfied by an equilibrium in the continuous type case. The conditions are exactly the same as in the limiting finite type case. This is satisfactory, because models with continuous types and models with finitely many types are theoretical tools for analyzing the same kind of real world problems. Put differently, it would be highly unsatisfactory if an equilibrium refinement concept applies to one case but not the other, or gives different answers for the two cases.

The paper is structured as follows. The next section presents the basic model. Section 3 presents examples and introduces the Local Credibility Test. Then Section 4 presents the formal model and, for the Spence model, provides a general characterization of the equilibrium satisfying the LCT. We derive conditions under which the LCT is satisfied by the candidate separating equilibrium in a limiting many type case. Section 5 extends the model to allow for different outside opportunities. Section 6 examines a general signaling model and shows that the conditions for the LCT to hold for a type that is separated are exactly the same for the continuum as for the limit of the finite type model. Concluding remarks are in Section 7.

2. The Model

We consider the following signaling environment. A player, the sender, has a signaling cost characteristic $s_i \in S = \{s_1, s_2, ...s_n\}$ and a value characteristic $v_j \in V = \{v_1, v_2, ...v_m\}$ valued by a receiver, where $s_i < s_{i+1}$ and $v_j < v_{j+1}$. The joint probability distribution of signaling cost and value characteristic is given by $\pi(s_i, v_j) \geq 0,\ i = 1,...,n,\ j = 1,...,m$, where $\sum_{i,j} \pi(s_i, v_j) = 1$. Let the unconditional probability distribution of s be $f(s_i)$, where $\sum_i f(s_i) = 1$, and let $F(\cdot)$ denote the associated cumulative distribution function ($F(s_i) = \sum_{i' \leq i} f(s_{i'})$). We assume that the two characteristics are negatively affiliated. Thus, the conditional expectation $v(s) = E[v \mid s]$ is a strictly increasing function. Extending this definition to all subsets of S we define $v(\hat{S}) = E(v(s) \mid s \in \hat{S})$. The standard signaling model usually considers the special case in which the negative correlation between s and v is perfect.

The sender knows her personal characteristic vector and chooses an action, $y \in \{\phi\} \cup Y$, where $Y = [\underline{y}, \infty)$ is the set of feasible signals. If the sender chooses not to signal, $y = \phi$. The receiver chooses an action x and the resulting utility of the sender is $U(s, x, y)$. We assume that utility is increasing in both t and x. We also assume that the single crossing property holds.

$$\left.\frac{dx}{dy}\right|_U = -\frac{\frac{\partial U}{\partial y}}{\frac{\partial U}{\partial x}} \text{ is a decreasing function of } s$$

The receiver does not know the characteristic vector of the sender but knows the distribution $\pi(s_i, v_j)$. He observes the sender's action y and forms a belief about which subset of types $\hat{S}$ has taken this action. We assume that the best response $x = r(s)$ is a smoothly increasing function of the point estimate s. We also assume that the receiver's payoff is linear in v so that it is only the perceived expected value of v which determines the receiver's response. Then we can write the receiver's best response as $x = r(\hat{v}),$ where $\hat{v} = v(\hat{S}) = E\{v(s) \mid s \in \hat{S}\}$. The sender's payoff is therefore $U(s, r(\hat{v}), y)$.

It proves helpful to transform variables and define the sender's "type" to be $t = v(s)$. That is, a sender's type is her true value to the receiver. For any subset of possible signaling cost characteristics, $\hat{S}$ there is a corresponding subset of types $\hat{T} = \{v(s) \mid s \in \hat{S}\}$. Define $\hat{t} = E\{t \mid t \in \hat{T}\}$. Since $v(s)$ is strictly increasing we can rewrite the sender's utility as follows.

$$u(t,\hat{t},y) = U(v^{-1}(t), r(\hat{t}), y)$$

Consider the following example first analyzed by Spence. A consultant with productivity v and signaling cost parameter s can signal at level y at a cost $c(s,y)$. Her expected productivity conditional upon s is $v(s)$ and her expected productivity given the belief that $s \in \hat{S}$ is $E\{v(s) \mid s \in \hat{S}\}$. Firms compete for her services and bid her wage up to her productivity thus $r(\hat{S}) = E\{v(s) \mid s \in \hat{S}\}$. The consultant's expected payoff is therefore

$$U(s, r(v(\hat{S})), y) = E\{v(s) \mid s \in \hat{S}\} - c(s,y).$$

Transforming variables, the consultant's type is her expected productivity and her expected payoff is

$$u(t,\hat{t},y) = \hat{t} - C(t,y), \text{ where } C(t,y) = c(v^{-1}(t), y) \text{ and } \hat{t} = E\{t \mid t \in \hat{T}\}.$$

The single crossing property holds if and only if the marginal cost of signaling is a decreasing function of the seller's type.

3. The Local Credibility Test

Consider the Spence consultant model discussed above. Initially we assume that each consultant observes her own signaling cost but not her productivity. There is a continuum of separating Nash equilibria in this signaling game. A separating Nash equilibrium with three signaling cost levels is depicted in Figure 3.1.

Each curve is an indifference curve for some signaling cost. A less heavy curve indicates a lower signaling cost. In a separating equilibrium, the market can infer the consultant's expected productivity t_i from the signal she sends. And thus pays her a wage equal to t_i. Note that the equilibrium choice for each t_i (indicated by a shaded dot) is preferred over the choice of the other types.

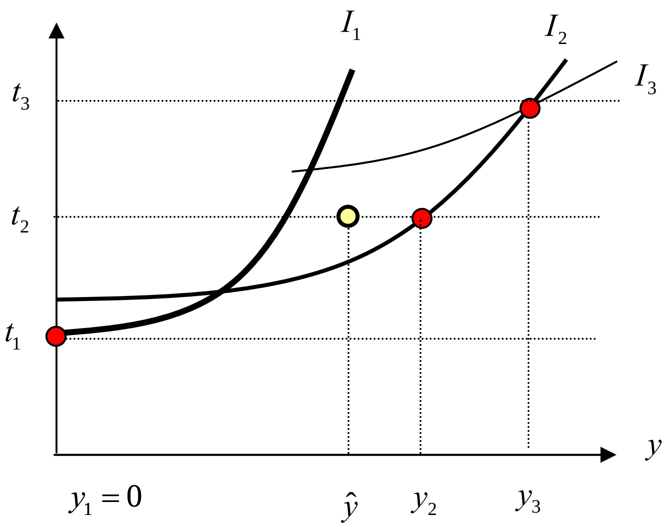

Fig: 3.1: Separating Nash Equilibria

Such an equilibrium fails the Intuitive Criterion proposed by Cho and Kreps (1987).[2] To see this, suppose a consultant chooses the signal $\hat{y}$ and argues that she is type s_2. Is this credible? If the consultant is believed, her wage will be bid up to $t_2 = v(s_2)$ so she earns the same wage as in the separating equilibrium but incurs a lower signaling cost. Moreover, $(\hat{y}, t_2)$ is strictly worse than (y_1, t_1) for type t_1 and strictly worse than (y_3, t_3) for type t_3. Thus the claim is indeed credible, hence this separating equilibrium does not survive the Intuitive Criterion.

Similar arguments rule out any Nash equilibrium where different signaling cost types are pooled. Thus the only equilibrium that satisfies the Intuitive Criterion is the *Pareto dominant separating equilibrium* (i.e., the Riley outcome) in which the lowest type chooses the smallest signal ($y = 0$) and each "local upward incentive constraint" is binding.

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[2] As noted by Cho and Kreps (1987), with more than two types, it is necessary to modify their original Intuitive Criterion or it loses much of its power. For the modified Intuitive Criterion the question is whether any particular type is uniquely able to benefit from some out-of-equilibrium signal if the signal receivers correctly infer the signaler's type.

Next suppose that each consultant knows both her signaling cost characteristic and her marginal product. Again consider the separating Nash Equilibrium depicted above. Suppose in this equilibrium three different types are pooled at each signal level. Consider the sellers with characteristic vectors three types (s_2, v_1), (s_2, v_2), (s_2, v_3) pooled at y_2 with the expected marginal product $t_2 = v(s_2)$. Suppose a consultant chooses $\hat{y}$ and claims to have characteristic vector (s_2, v_3). Is this credible? If the claim is believed, the consultant's wage will rise from $t_2 = v(s_2)$ to v_3 thus the consultant is indeed better off. But any offer that makes type (s_2, v_3) better off also make types (s_2, v_1) and (s_2, v_2) better off, since they have the same signaling cost. Thus there is no credible claim that type (s_2, v_3) *alone* can make. A similar argument holds for each of the other types. Thus any Nash separating equilibrium satisfies the Intuitive Criterion. An almost identical argument establishes that any Nash Equilibrium with (partial) pooling satisfies the Intuitive Criterion as well.

Since all the consultants with the same signaling cost characteristic are observationally equivalent, it seems to us that any argument for ranking the equilibria in the first model (productivity unknown) should also be applicable to the second model (productivity known) as well. The discussion also makes clear that a solution that achieves this goal should allow the possibility of pooling deviations in addition to deviations by those with a particular characteristic vector. That is, if a pool of senders with different characteristics can credibly deviate to an out of equilibrium signal so that they can be better off while those with other characteristic vectors cannot, then the equilibrium fails the refinement test. In the above example, if an out of equilibrium signal $\hat{y}$ is observed, the receiver should allow the possibility that the sender can have any of the three characteristic vectors (s_2, v_1), (s_2, v_2), (s_2, v_3). The question is what belief should the receiver have? Consistent with Cho and Kreps' original idea, one way to generalize their Intuitive Criterion (while allowing pooling deviations) is to suppose that the receiver has the most conservative belief that the sender is the lowest type from the pool. However, this generalization does not have power in the above example, because it is not the case that those with characteristic vectors $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$ would all be better off deviating to $\hat{y}$ if the receiver's belief is v_1.[3]

Given that upon observing the out of equilibrium signal $\hat{y}$ the receiver thinks that it can be any of the three types $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$, it is natural that she uses the Bayes Rule so her expected marginal product should be $t_2 = v(s_2)$. Under this belief, a deviation to $\hat{y}$ by the pool of $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$ is clearly credible: any type in this pool is better off but types not in the pool are worse off from such a deviation. Then once again, the unique Nash Equilibrium satisfying this refinement test is the Pareto Dominant separating equilibrium.

We now introduce the formal definition of the Local Credibility Test.

Let $\hat{y}$ be an out-of-equilibrium signal and suppose that y^- is the largest Nash Equilibrium signal less than $\hat{y}$ (if any.) Let S^- be the subset of types choosing y^-. Similarly let y^+ be the smallest Nash Equilibrium signal greater than $\hat{y}$ (if any) and let S^+ be the subset of signaling cost types choosing y^+. Define $\hat{S} = S^- \bigcup S^+$. Finally define $S^L = \{S^-, S^+, \hat{S}\}$ and the expected values $\overline{v}(S) = E\{v \mid S \in S^L\}$.

Local Credibility Test (LCT):

A Nash Equilibrium passes the Local Credibility Test (LCT) if there is no $\hat{y}$ and associated $S \in S^L$ such that $(\hat{y}, \overline{v}(S))$ is strictly preferred by type s over the Nash Equilibrium outcome if and only if $s \in S$.

For strong local credibility we again consider the local types in $\hat{S} = S^- \bigcup S^+$ but require that no subset of types can credibly defect.

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[3] Similarly, it can be verified that the Cho and Sobel (1990)'s refinement concept of "divinity", which is built on the idea of stability of Kohlberg and Mertens (1986) and can be considered as a logic offspring of the Intuitive Criterion, does not have power either in the above example. Ramey (1996) extends the Cho and Sobel's divinity concept to the case of a continuum of types. Like the Intuitive Criterion, divinity faces the same problem of distinguishing types $(s_2, v_1),\ (s_2, v_2),\ (s_2, v_3)$ to interpret a possible deviation, while these types have the same incentives to deviate. Riley (2001) discusses in greater details these and other refinement concepts.

Strong Local Credibility Test (LCT):

A Nash Equilibrium passes the Strong Local Credibility Test if there is no $\hat{y}$ and any subset of types $S \subset \hat{S}$ such that $(\hat{y}, \overline{v}(S))$ is strictly preferred by type s over the Nash Equilibrium outcome if and only if $s \in S$.

Heuristically, the sender who chooses the out of equilibrium signal $\hat{y}$ can make the following statement to the receiver: "I am in the subset S and you should believe me, because if you do and apply the Bayes Rule to update your belief, every type in S will be better off and all other types will be worse off than in the equilibrium." If there exists such a pair $(\hat{y}, S)$, the equilibrium fails the LCT.

Note that if $\hat{y}$ is smaller (greater) than all equilibrium signals, then y^- (y^+) does not exist and y^+ (y^-) is the smallest (largest) equilibrium signal. By the above definition, $\hat{S}$ is the subset of types choosing y^+ (y^-)). Also note that by considering a subset of $\hat{S}$ to be the singleton set of a single type choosing y^+ or y^- , the definition of the Strong Local Credibility Test allows deviations by single types. It follows that the strong LCT test is stronger than the Intuitive Criterion[4] and hence only the Pareto Dominant separating equilibrium can pass the LCT.

On the other hand, the idea of the Strong Local Credibility Test is weaker than the refinement concept proposed by Grossman and Perry (1986a,b) in bargaining models. For any out-of-equilibrium signal $\hat{y}$, their criterion considers *any* subset of types as a potential deviating pool. An equilibrium fails the Grossman and Perry test if $\hat{y}$ is credible for one subset of types. In signaling models the Grossman and Perry test is often too strong because no equilibrium can pass the test, especially when the type space is large. Here we restrict attention to local deviations. This makes the analysis more tractable and, we believe, more plausible: when rational players experiment with deviations, they are more likely to experiment with small than with large deviations;

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[4] To be precise, the Intuitive Criterion is only used locally hence a "local" Intuitive Criterion has the same power as the global version. The LCT is a more powerful test than the local Intuitive Criterion.

whatever gives rise to “unsent” signals is likely to give rise to signals near those that are meant to be sent.

One simple way to make our concept “local” operational is as follows. We assume that for each type t_i, he chooses the equilibrium signal $y(t_i)$ with probability $1-\rho$, and trembles with probability ρ. When he trembles, he trembles to $[y(t_{i-1}), y(t_{i+1})]$ uniformly. For the lowest type the trembling interval is $[0, y(t_2)]$; Analogously, for the highest type the trembling interval is $[y(t_{n-1}), y_{\max}]$. With this behavioral structure, any out-of-equilibrium signal should come from “nearby” types, which is similar our “local” idea.

More importantly, we now argue that the Pareto Dominant separating equilibrium survives the LCT test only when it makes sense. To see that the Pareto Dominant separating equilibrium can sometimes defy common sense, consider the following example. Suppose there are two signaling cost types. For those with a high signaling cost ($s = s_1$), the cost of signaling is $c_1(y)$ with $c_1(0) = 0$ and $c_1' > 0$, and the expected marginal product is 100. For those with a low signaling cost $(s = s_2)$, the signaling cost is $c_2(y) = (1-\frac{\varepsilon}{100})c_1(y)$ and the expected marginal product is 200. The Pareto dominant separating equilibrium is depicted below in Figure 3.2.

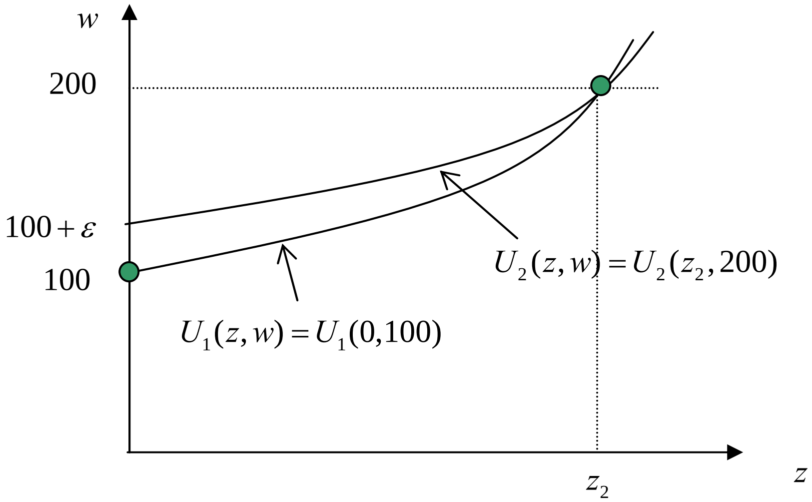

Fig. 3.2: Separating equilibrium with a gain of ε

The low type must be indifferent between $(0, v(s_1))$ and the choice of type s_2, that is $(s_2, v(s_2))$. Therefore,

$$U_1 = 100 - 200 - c_1(y_2) \text{ and so } c_1(y_2) = 100 .$$

The payoff for type s_2 is therefore

$$U_2 = v(s_2) - c_2(y_2) = 200 - (1 - \frac{\varepsilon}{100})c_1(y_2) = 100 + \varepsilon .$$

Suppose that only 1 in 100 consultants is of type s_1. Then the unconditional mean marginal product is 199. Thus essentially all the social surplus generated by the high types is dissipated by signaling and both types have an income which is approximately half the income they would have in the Nash pooling equilibrium! We believe a good criterion for ranking equilibria should not rule out pooling equilibria in such circumstances. Applying the LCT test, it is easy to see that the Pareto dominant separating equilibrium does not survive the test. Consider a pooling deviation that both types deviate to an out of equilibrium signal $\hat{y}$ sufficiently close to zero. Then they both will be better off since the expected marginal product (and hence the wage) is 199 while the signaling cost is very small. Next consider the pooling equilibrium in which both types choose $y = 0$. Consider any out-of-equilibrium signal $\hat{y} > 0$. Then S^- is the entire set of types. The most firms will pay for this type is the average productivity. Thus no one is better off choosing $\hat{y}$ and so the pooling equilibrium satisfies the LCT.

In this simple example, it is easy to determine when the Pareto dominant separating equilibrium also passes the LCT. For concreteness, suppose $c_1(y) = 100y$, $c_2(y) = (100 - \varepsilon)y$, and the probability of a consultant being type s_1 is q. It can be verified that the equilibrium survives the LCT if and only if $q > 1 - 0.01\varepsilon$.[5] Therefore, the

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[5] In the equilibrium, $y_1 = 0$, $y_2 = 1$, $U_1 = 100$ and $U_2 = 100 + \varepsilon$. By the definition of the LCT, consider $S = \hat{S} = \{s_1, s_2\}$. The average productivity of these two signaling cost types is $\overline{v} = 200 - 100q$. For $\hat{y} = \lambda \in (0,1)$ to be a credible deviation by S, both types must strictly prefer $(\hat{y}, \breve{v})$. Note that $U_1(\hat{z}, \overline{v}) = 100 - 100q - (100 - \varepsilon)\lambda$ and $U_2(\hat{z}, \overline{v}) = 200 - 100q - (100 - \varepsilon)\lambda$.

larger the proportion of the low type, or the greater the marginal cost difference between types (i.e., the stronger the signal), the more likely the Pareto dominant separating equilibrium passes the LCT test.[6]

Another advantage of the LCT is that it can be applied consistently in both the finite type and the continuous type cases. To illustrate this, we now show in a simple Spence education signaling model how the LCT can be applied when there are many types and, in the limit, a continuum of types. Suppose type t_i has a signaling cost $C(t_i, y) = \frac{y}{H(t_i)}$. The utility of a type t_i worker perceived to be type $\hat{t}$ is then

$$u(t_i, \hat{t}, y) = \hat{t} - \frac{y}{H(t_i)}.$$

We seek conditions under which the Pareto dominant separating equilibrium passes the LCT. In this equilibrium, the local upward incentive constraints are binding. Consider the figure below.

An indifference line for type *t* is labeled $I(t)$. Those with signaling cost type t_{i-1} are indifferent between their separating outcome $(y(t_{i-1}), t_{i-1})$ and $(y(t_i), t_i)$. Similarly, both $(y(t_i), t_i)$ and $(y(t_{i+1}), t_{i+1})$ are on the indifference line $I(t_i)$. Consider all alternatives that are preferred by types t_i and t_{i+1} but not by type t_{i-1} and which are also profitable. These are shaded in the figure.

The distance EK$= \frac{f_{i+1}}{f_i + f_{i+1}} \Delta$ where $\Delta = t_{i+1} - t_i$. Since the slope of $I(t_{i-1}) = 1/H(t_{i-1})$, the distance

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Thus, if there exists a λ such that $U_1(\hat{z}, \overline{v}) = 200 - 100q - 100\lambda > 100 = U_1$ and $U_2(\hat{z}, \overline{v}) = 200 - 100q - (100 - \varepsilon)\lambda > 100 + \varepsilon = U_2$, then both types are indeed better off choosing the out-of-equilibrium signal $\hat{z} = \lambda$ and the equilibrium will fail the LCT test. For the equilibrium to satisfy the LCT, it must be that $q > 1 - 0.01\varepsilon$.

[6] Since there are no additional subsets, the equilibrium also satisfies the strong LCT.

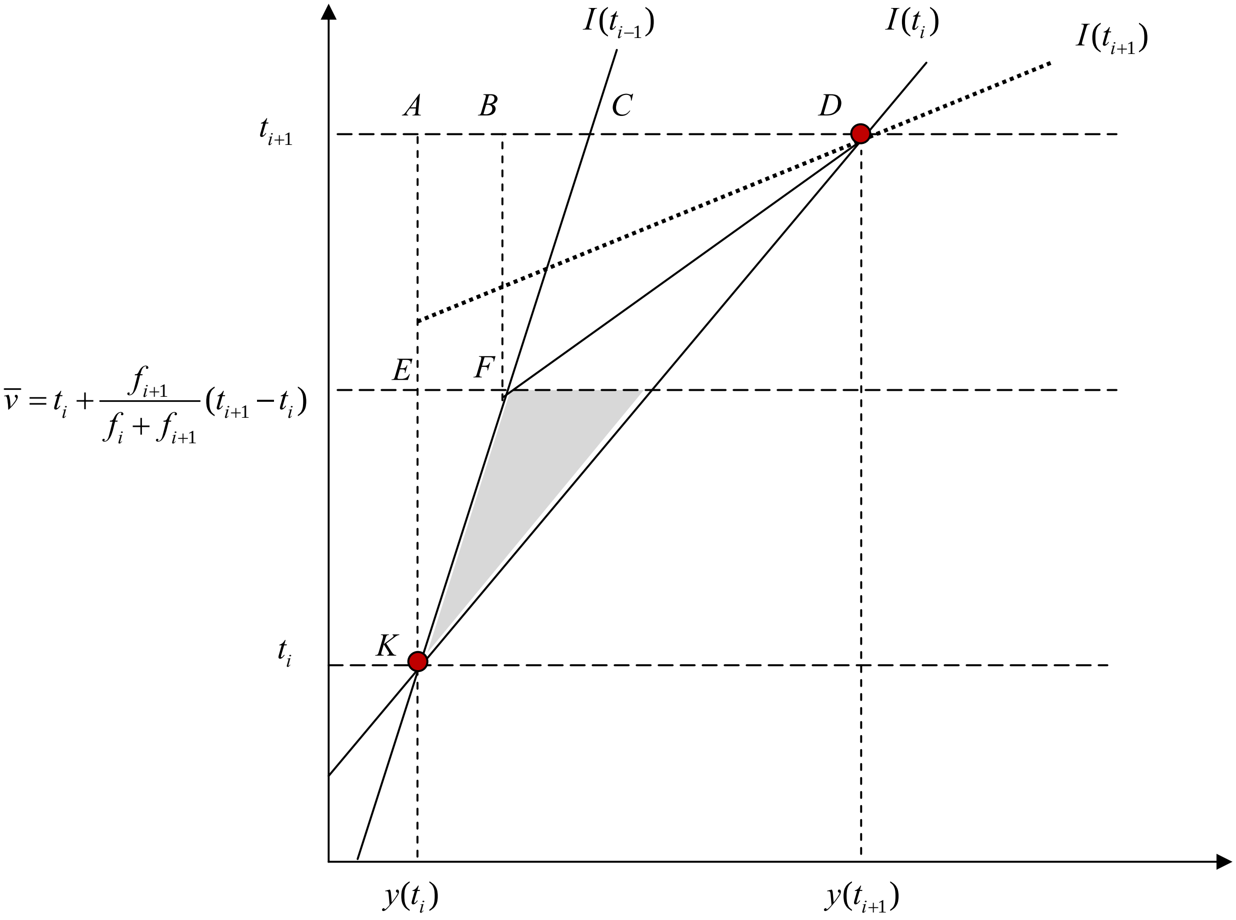

Fig. 3.3: Applying the LCT

AB $= \frac{f_{i+1}}{f_i + f_{i+1}} H(t_{i-1})$. By an almost identical argument, the distance AD $= H(t_i)\Delta$. Also

BF =AK-EK $= \frac{f_i}{f_i + f_{i+1}} \Delta$. Then the slope of the line FD is $\dfrac{\frac{f_i}{f_i + f_{i+1}}}{H(t_i) - \frac{f_i}{f_i + f_{i+1}} H(t_{i-1})}$.

If, as depicted, the dashed indifference line for type t_{i+1} does not intersect the interior of the shaded region, none of these shaded alternatives are incentive compatible for type t_{i+1} and so the local credibility test holds. If the indifference line does intersect the shaded region the separating equilibrium fails the LCT. Thus the LCT holds for type t_i if and only if

$$\frac{\dfrac{f_i}{f_i + f_{i+1}}}{H(t_i) - \dfrac{f_i}{f_i + f_{i+1}} H(t_{i-1})} \geq \frac{1}{H(t_{i+1})}.$$

Rearranging this inequality, the LCT holds for type t_i if and only if

$$\frac{(H(t_{i+1}) - H(t_i) - (H(t_i) - H(t_{i-1}))}{H(t_i) - H(t_{i-1})} \geq \frac{f(t_{i+1}) - f(t_i)}{f(t_i)}.$$

Note that this condition is independent of the step size Δ between signaling cost types. Treating the continuum of types as the limit when $\Delta \to 0$, it follows that when types are sufficiently close, the Pareto dominant separating equilibrium passes the LCT at *t* if and only if

$$\frac{H''(t)}{H'(t)} \geq \frac{F''(t)}{F'(t)}.$$

This is exactly the same as in the Spence education signaling model with a continuum of types, which we discuss in Example 1 of Section 6.

4. Characterization of equilibria

Throughout we will make the following assumptions about the response by the receiver and the utility function of the sender $u(t, \hat{t}, y)$.

Assumption S (Single Crossing)

(S1) $r'(t) > 0$ (S2) $u_2 > 0$, (S3) $u_3 < 0$ (S4) $\frac{\partial}{\partial t}(-\frac{u_3}{u_2}) < 0$.

Condition S1 requires that the receiver's best response is a strictly increasing differentiable function of the sender's type. Then condition S2 holds as long as a bigger response by the receivers is strictly preferred by the sender. Condition S3 requires that the sender's action is costly. Condition S4 is the requirement that the marginal willingness to signal in order to induce a higher response is higher for higher types.

Lemma 4.1: Monotonicity of Nash equilibria

If Assumption S holds, then a Nash Equilibrium mapping from the sender's expected value to her signal, $y(t)$ is a weakly increasing function.

Proof: The proof follows in essence arguments made by Spence (1973). Graphically, suppose that type t_i chooses y_i. Then all other alternatives must lie on or below her indifference curve through the point $(y_i, \hat{t}_i)$, that is below the curve $I(t_i)$ in the figure. For any higher type t_j, the indifference curve $I(t_j)$ through this point is flatter. Type $t_j's$ choice must lie on or above this indifference curve. Combining these statements, type t_j's choice must lie in the shaded region. That is $y_j \geq y_i$.

QED

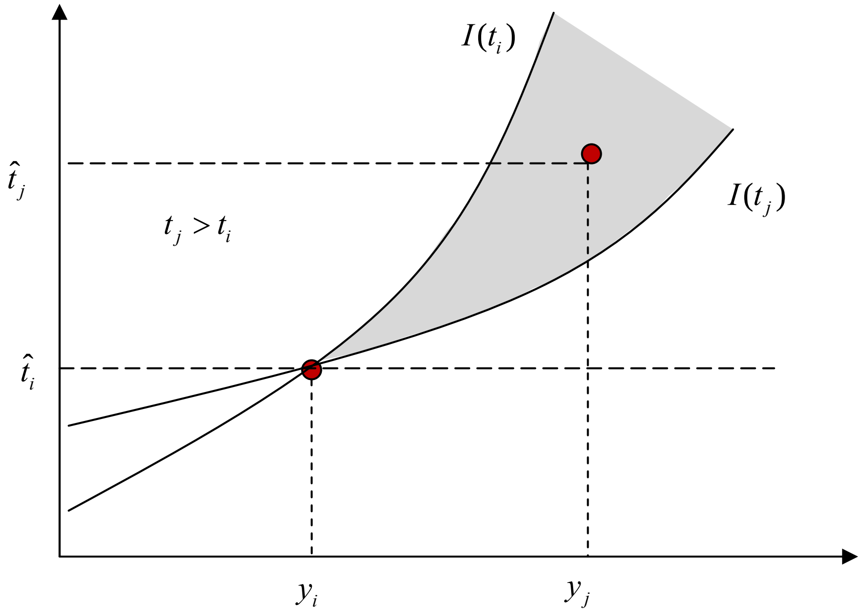

Fig. 4.1: Single Crossing and monotonicity

Proposition 4.2: To satisfy the LCT, the Nash equilibrium signal of the lowest type is zero.

Proof: Sketch.

Intuitively, suppose that $y(t_1) > 0$ and there is some subset of types choosing this signal. Let t_i be the highest type in the pool. Given Lemma 4.1, all types $t < t_i$ are in the pool. Then the utility of type *t* in the pool is $u(t, a(t_i), y(t_1))$ where $a(t_i) = E\{\tilde{t} \mid \tilde{t} \leq t_i\}$.

Consider the out-of-equilibrium signal $\hat{y} < y(t_1)$. Then $S^+ = \{t \mid t \leq t_i\}$. By the LCT, the response to $\hat{y}$ must be at least $a(t_i)$ since all types are better off with a lower signal ($u_3 < 0$).

QED

Proposition 4.3: Tightness Condition

For a Nash Equilibrium to survive the LCT, the local upward constraint must be binding for each type.

Proof: Suppose that the upward constraints are not binding for type t_i. Consider first the case when type t_i is separated. Then the equilibrium wage for this type is t_i. Since the single crossing property holds, the equilibrium mapping $y(t_j),\ j = 1.,,,n$ is weakly increasing. Let the equilibrium signal be y_i Given separation, type t_{i+1} must have a strictly higher signal y_{i+1}. Let S_{i+1} be the subset of types choosing y_{i+1} and define $\hat{t} = E\{t \mid t \in S_{i+1}\}$. This is the expected productivity of those choosing y_{i+1}. Suppose that the local upward constraints are binding for all $t > t_i$. We will show that to satisfy the LCT, the local upward constraint must be binding for t_i. Consider the figure below. By hypothesis the upward constraint is binding for type t_{i+1}. Since the slope of the indifference curve of type t_{i+1} is less than that for type t_i, type t_{i+1} (and all higher types) strictly prefer $(y_{i+1}, \hat{t})$ over (y_i, t_i). Suppose that the local upward constraint for type t_{i-1} is not binding. Then there is a region of signal wage pairs that is preferred only by type t_i and is profitable for all wages less than t_i. This is the shaded region in the figure. Then consider the out-of-equilibrium $\hat{y} = y_i - \delta$ where $\delta > 0$ is sufficiently small. By definition the local types to the right $S^+ = \{t_i\}$. If offered a wage t_i type t_i are strictly

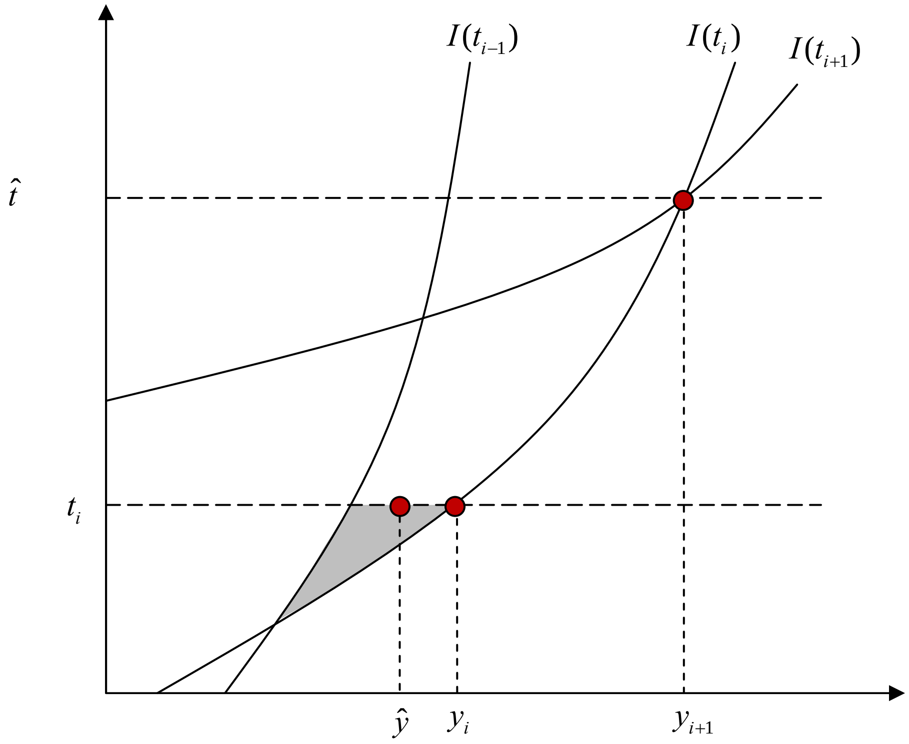

Fig. 4.2: Local upward constraint not binding

better off and all other types are strictly worse off. Then the equilibrium fails the LCT for S^+.

To complete the inductive argument we need to establish that the local upward constraint must be binding for the highest type. But this follows immediately since for the highest type there are no downward constraints. Finally we note that an almost identical argument holds if there are a pool of types choosing $y(t_i)$.

QED

Proposition 4.4: If assumption S holds and the slope of $a(t) = E\{\tilde{t} \mid \tilde{t} \le t\}$ is strictly positive at t_0, all those types sufficiently close to the lowest type must be pooled in order to satisfy the LCT.[7]

Proof: Suppose that types t_1 and t_2 are separated. By Proposition 4.3, type t_1 must be indifferent between his signal and the signal of type t_2. Then

$$u(t_1, t_1, y(t_1)) = u(t_1, t_2, y(t_2)).$$

Let $U^S(t)$ be the equilibrium utility of type *t*. Then

$$\begin{aligned} U^S(t_2) - U^S(t_1) &= u(t_2, t_2, y(t_2)) - u(t_1, t_1, y(t_1)) \\ &= u(t_2, t_2, y(t_2)) - u(t_1, t_2, y(t_2)) \end{aligned}$$

Hence

$$\frac{U^S(t_2) - U^S(t_1)}{t_2 - t_1} = \frac{u(t_2, t_2, y(t_2)) - u(t_1, t_1, y(t_1))}{t_2 - t_1}.$$

Taking the limit as $t_2 \to t_1$

$$\lim_{t_2 \downarrow t_1} \frac{U^S(t_2) - U^S(t_1)}{t_2 - t_1} = u_1(t_1, t_1, 0).$$

Suppose instead that types t_i and all lower types pool. Then

$$U^P(t_i) = u(t_i, a(t_i), 0)$$

And so

$$U^P(t_2) - U^P(t_1) = u(t_2, r(a(t_2)), 0) - u(t_1, a(t_1), 0).$$

Dividing by $t_2 - t_1$ and taking the limit,

$$\lim_{t_2 \downarrow t_1} \frac{U^P(t_2) - U^P(t_1)}{t_2 - t_1} = u_1(t_1, t_1, 0) + u_2 a'(t).$$

Thus for t_2 sufficiently close to t_1, $U^P(t_2) > U^S(t_2)$.

---

[7] In the limiting continuous case, where $t_1 \to t_0$ sufficient conditions for $a'(t_0) > 0$ are either (i) $F'(t_0) > 0$, or (ii) $F'(t_0) = 0$ and $F''(t_0) > 0$

Next apply the LCT by considering a signal $\hat{z} \in (z(t_1), z(t_{i+1}) = (z(t_i), z(t_{i+1})$. The argument parallels that in the proof of Proposition 4.2.

QED

We now provide a complete characterization of Nash Equilibria that satisfy the LCT. To keep the analysis as clear as possible we focus on the Spence labor market model. That is, if the perceived type is $\hat{t}$

$$u(t, \hat{t}, y) = \hat{t} - C(t, y).$$

To further simplify the analysis we consider the special case of a multiplicatively separable cost,

$$C(t, y) = y / H(t),$$

where $H(\cdot)$ is strictly positive and strictly increasing. From section 3, a Nash equilibrium that is separating at *t* satisfies the LCT if and only if

$$\frac{H''(t)}{H'(t)} \geq \frac{F''(t)}{F'(t)}. \tag{4.1}$$

As both F and H are increasing functions, there exists an increasing function $p(\cdot)$ such that $H(t) = p(F(t))$. From the theory of risk aversion we know that (4.1) holds if and only if the function $p(\cdot)$ is convex. Thus the necessary and sufficient condition for the LCT to hold at *t* is that *H* is more convex than *F* at *t*.

Since it simplifies the mathematics, we consider only the limiting case when the number or types approaches a continuum, that is, $t_1 \downarrow t_0$, $t_n \uparrow t_\infty$ and, in the limit, $t \in [t_0, t_\infty]$. Of course the choice of the support is completely arbitrary.

Appealing to the tightness property, if there is separating in the neighborhood of type t_i, this type is indifferent between her signal and that of type t_{i+1}. That is.

$$u(t_i, t_{i+1}, y(t_{i+1})) - u(t_i, t_i, y(t_i)) = 0.$$

Hence

$$\frac{u(t_i, t_{i+1}, y(t_{i+1})) - u(t_i, t_i, y(t_{i+1}))}{t_{i+1} - t_i} + \frac{u(t_i, t_i, y(t_{i+1})) - u(t_i, t_i, y(t_i))}{t_{i+1} - t_i} = 0$$

Taking the limit,

$$u_2 + u_3 y'(t) = 0 \qquad \text{FOC}$$

For the Spence model $u(t,\hat{t},y)=\hat{t}-C(t,y)$. Then the FOC becomes

$$1-C_2(t,y)y'(t)=0\,.$$

If all types in $[0,t]$ pool, the wage is $a(t)=E\{t\,|\,\tilde{t}\le t\}$. Thus

$$U^P(t)=a(t)=\frac{\int_0^t xdF(x)}{F(t)}\,. \tag{4.2}$$

Suppose instead that all types in some interval $[\tau,\hat{\tau}]$ are separated. Then on this interval

$$U^S(t)=t-C(t,y(t))=t-\frac{y(t)}{H(t)} \tag{4.3}$$

Differentiating by t

$$\frac{d}{dt}U^S(t)=u_1+u_2+u_3y'(t)\,.$$

By the FOC the sum of the second and third terms is zero. (This is the Envelope Theorem.) From (4.3) if follows that

$$\frac{d}{dt}U^S(t)=\frac{H'(t)}{H(t)}\frac{y(t)}{H(t)}\,. \tag{4.4}$$

Proposition 4.5: Family of solutions to the differential equation

There exists a family of solutions $U^S(t\,|\,k)$ to the differential equation (4.4). Moreover there exists k^* and τ^* such that (i) $U^S(\tau^*\,|\,k^*)=U^P(\tau^*)$,

(ii) $\frac{d}{dt}U^S(\tau^*\,|\,k^*)=\frac{d}{dt}U^P(\tau^*)$ and (iii) $U^S(t\,|\,k^*)>U^P(t),\ t\in[\tau^*,t_\infty]$

Proof: While this proposition holds generally[8] we present a proof for the Spence model. Substituting from (4.3),

$$\frac{d}{dt}U^S(t)=\frac{H'(t)}{H(t)}(t-U^S(t))\,,\quad t\in[\tau,\tau'] \tag{4.5}$$

Rearranging the differential equation,

$$H(t)\frac{d}{dt}U^S(t)+H'(t)U^S(t)=tH'(t)\,.$$

Reintegrating,

---

[8] The full proof follows from an extension of the analysis by Riley (1980)

$$h(t)U^S(t \mid k) = \int_{t_0}^{t} tH'(t)dt + k \;.$$

Thus

$$U^S(t \mid k) = \frac{\int_{t_0}^{t} tH'(t)dt + k}{H(t)} \;.$$

Note that $U^S(t \mid k)$ is strictly increasing in *k*. Also, since $H'(\cdot) > 0$ it follows that the slope of $U^S(t \mid \tau)$ is strictly decreasing in *k*. Let k_0 be the value of *k* such that $U^S(t_0 \mid k_0) = t_0$. This is the Nash Equilibrium in which every type is separated. From Proposition 4.2 the lowest type chooses $y(t_0) = 0$ Thus, from (4.4),, $U^S(t \mid k_0)$ is strictly increasing and the slope is zero at t_0. This curve is depicted below.

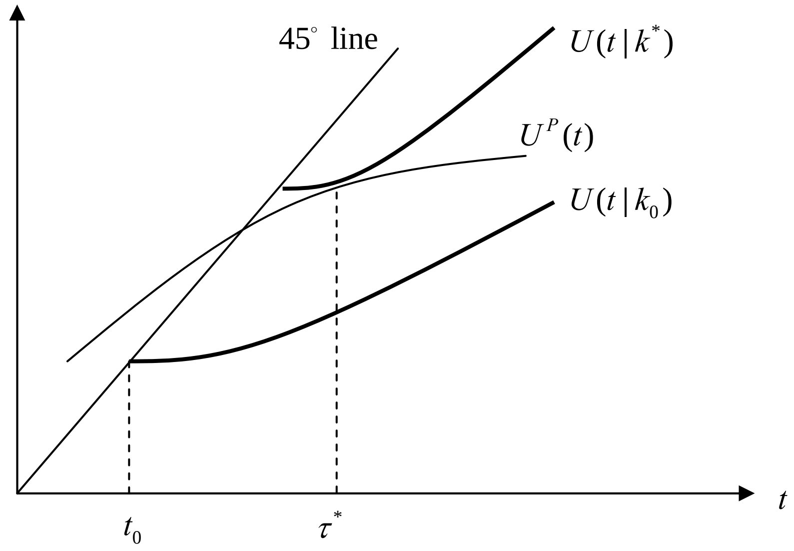

Fig. 4.3: Family of solutions

For small *t* this lies below $U^P(t)$. Hence there is some $k^* > k_0$ and type τ^* such that (i) $U^S(\tau^* \mid k^*) = U^P(\tau^*)$, (ii) $\frac{d}{dt}U^S(\tau^* \mid k^*) = \frac{d}{dt}U^P(\tau^*)$ and

(iii) $U^S(t \mid k^*) > U^P(t), \; t \in [\tau^*, t_\infty]$

The next proposition establishes that the more rapidly that the marginal cost of signaling declines with type, the greater is the signaling equilibrium payoff.

Proposition 4.6: Weak and strong signals

Let $U_i(t),\ i=1,2$ be equilibrium payoffs when the cost of signaling is $C_i(t,y)=y/H_i(y)$. Suppose that types signal if and only if $t\geq\tau$.

If $\dfrac{H_1'(t)}{H_1(t)}>\dfrac{H_2'(t)}{H_2(t)},\ t>\tau$ then $U_1(t)>U_2(t),\ t>\tau$.

Proof: See Appendix

We are now ready to characterize equilibria satisfying the LCT.

Proposition 4.7: Weak Signals and Pooling

If $H'(t)/H(t)<F'(t)/F(t),\ t\in[0,1]$ then the unique Nash Equilibrium that satisfies the LCT is the pooling equilibrium.

Proof: By Proposition 4.4 we know that there is pooling for types sufficiently close to t_0. Then over this interval the sellers payoff is $U^P(t)$. Differentiating (4.2) by *t* and rearranging,

$$\frac{dU^P}{dt}=\frac{F'(t)}{F(t)}(t-U^P(t)). \tag{4.6}$$

We then appeal to conditions (4.6) and (4.5) to characterize equilibria satisfying the LCT. Suppose that there is an interval $[\tau,\tau']$ over which types separate. For the highest type in the pool the equilibrium payoff is $U^P(\tau)$. This is also the lowest type that is separated.

This type must be indifferent between pooling and separating. Thus the lowest signal $y(\tau)$ must satisfy

$$U^P(\tau) = U^S(\tau) = \tau - y(\tau)/h(\tau).$$

Appealing to (4.6) and the fact that $U^P(\tau) = U^S(\tau)$,

$$\frac{dU^P}{dt} = \frac{F'(t)}{F(t)}(t - U^S(t)).$$

Also, appealing to (4.5)

$$\frac{d}{dt}U^S(t) = \frac{H'(t)}{H(t)}(t - U^S(t)).$$

By hypothesis $\frac{H'(t)}{H(t)} < \frac{F'(t)}{F(t)}$ for all *t*. Hence $\frac{d}{dt}U^S(\tau) < \frac{d}{dt}U^P(\tau)$. Since $U^S(\tau) = U^P(\tau)$ it follows that there exists some interval (τ, τ') over which $U^S(t) < U^P(t)$. Then arguing as in the proof of Proposition 4.2, the equilibrium fails the LCT. Thus there can be no interval with types separated.

QED

To characterize equilibria with signaling, we will appeal to the following lemma.

Lemma 4.8: If for some τ, $\frac{H'(\tau)}{H(\tau)} > \frac{F'(\tau)}{F(\tau)}$ and for all $\tau > \tau$, $\frac{H''(t)}{H'(t)} > \frac{F''(t)}{F'(t)}$, then $\frac{H'(\tau)}{H(\tau)} > \frac{F'(\tau)}{F(\tau)}$ for all $t > \tau$.

Proof: See appendix

Proposition 4.9: **Strong signals and signaling equilibria**

Suppose (1) Assumption S holds, (2) $H(t_0) > 0$, (3) $H'(t_\infty)/H(t_\infty) > F'(t_\infty)$ and (4) the necessary and sufficient condition for a separating equilibrium to satisfy the LCT holds for all *t*. Then, there exists $\tau^* \in (t_0, t_\infty)$ such that for any $\tau \geq \tau^*$ there is a Nash Equilibrium satisfying the LCT with pooling of all types below τ and separation of types for all higher t. The higher is τ the greater the utility of the low types and the lower the utility of all those that separate.

Proof: Define $U^S(t \mid k(\tau))$ to be the Nash Equilibrium payoffs when those below τ pool and the rest separate. By (iii) $H'(t_\infty)/H(t_\infty) > F'(t_\infty)/F(t_\infty)$. Since $F(t_0)=0$, for all t sufficiently small, $H'(t)/H(t) < F'(t)/F(t)$ Thus there is some τ^* such that $H'(\tau^*)/H(\tau^*) = F'(\tau^*)/F(\tau^*)$. From Proposition 4.5 there exists some k^* and τ^* such that (i) $U^S(\tau^* \mid k^*) = U^P(\tau^*)$, (ii) $\frac{d}{dt}U^S(\tau^* \mid k^*) = \frac{d}{dt}U^P(\tau^*)$ and (iii)
$U^S(t,k^*) > U^P(t),\ t \in [\tau^*, t_\infty]$
This is depicted below.[9]

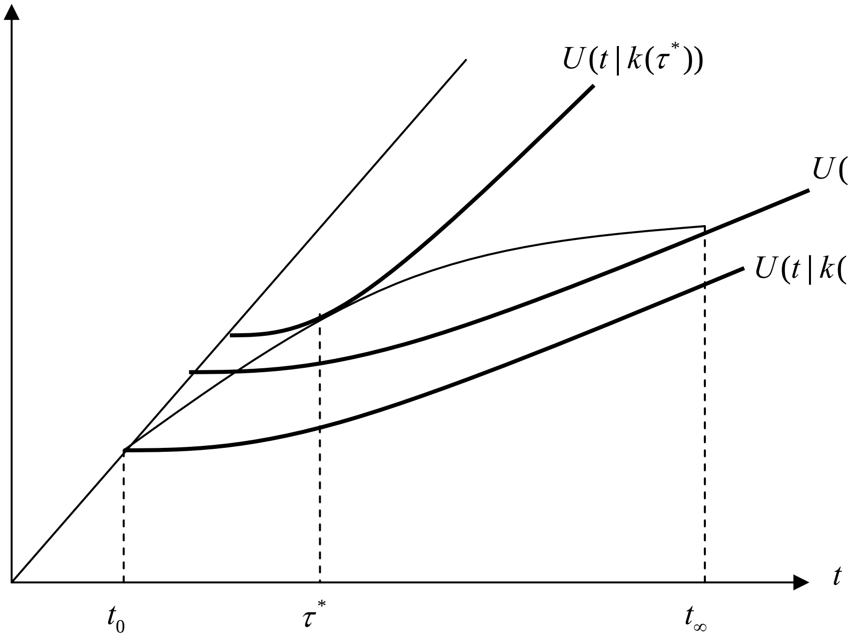

Fig. 4.4: Family of solutions

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[9] As depicted $k(t_0) < k(t_\infty)$. This need not be the case.

From (4.5) and (4.6)

$$\frac{d}{dt}U^S(t) = \frac{H'(t)}{H(t)}(t - U^S(t)) \text{ and } \frac{dU^P}{dt} = \frac{F'(t)}{F(t)}(t - U^P(t))$$

In some right neighborhood $[\tau^*, \tau^* + \delta]$, $\frac{d}{dt}U^S(t) \geq \frac{d}{dt}U^P(t)$, since $U^S(t) > U^P(t),\ t > \tau^*$.. Hence over this interval $\frac{H'(t)}{H(t)}(t - U^S(t)) \geq \frac{F'(t)}{F(t)}(t - U^P(t))$.

Also over this interval, since $U^S(t) > U^P(t)$, it follows that

$$\frac{H'(t)}{H(t)} > \frac{F'(t)}{F(t)},\ t \in (\tau^*, \tau^* + \delta]$$

Then if the necessary and sufficient condition for the LCT to hold at a point of separation holds for all $t \geq \tau^*$, it follows from Lemma 4.8 that

$$\frac{H'(t)}{H(t)} > \frac{F'(t)}{F(t)},\ t \in (\tau^*, t_\infty] \tag{4.7}$$

Then, arguing as in the proof of the previous proposition, it follows from inequality (4.7) that if $\tau \geq \tau^*$and $U^P(\tau) = U^S(\tau)$, then for some right neighborhood of τ, $U^P(t) < U^S(t)$. Thus there can be at most one intersection.

It follows that there are a family of signaling equilibria in which all those type less than $\tau \geq \tau^*$ are pooled and all higher types are separated.

QED

5. The Strong Local Credibility Test

In the LCT receivers consider only the pools of neighboring types. Under the strong credibility test (the local version of the Grossman Perry SPNE) we consider deviations by all subsets of those in nearby pools. The critical deviation is by the highest type in a pool since she has the lowest signaling cost and thus the greatest incentive to deviate.[10]

Proposition 5.1: A Nash Equilibrium satisfying the LCT also satisfies the strong LCT if and only there are no pools.

<u>Proof</u>: Let t_i be the highest type in a pool and let $\hat{t}$ be the expected value of those types that are pooled. Then $y(t_{i+1}) > y(t_i)$. Let $\hat{\hat{t}}$ be the expected value of those types choosing $y(t_{i+1})$. The equilibrium is depicted below.

The shaded region is preferred by type t_i alone. Also $\hat{t} < t_i < \hat{\hat{t}}$. Thus for $\hat{y}$ sufficiently close to $y(t_{i+1})$ the strong LCT fails.

QED

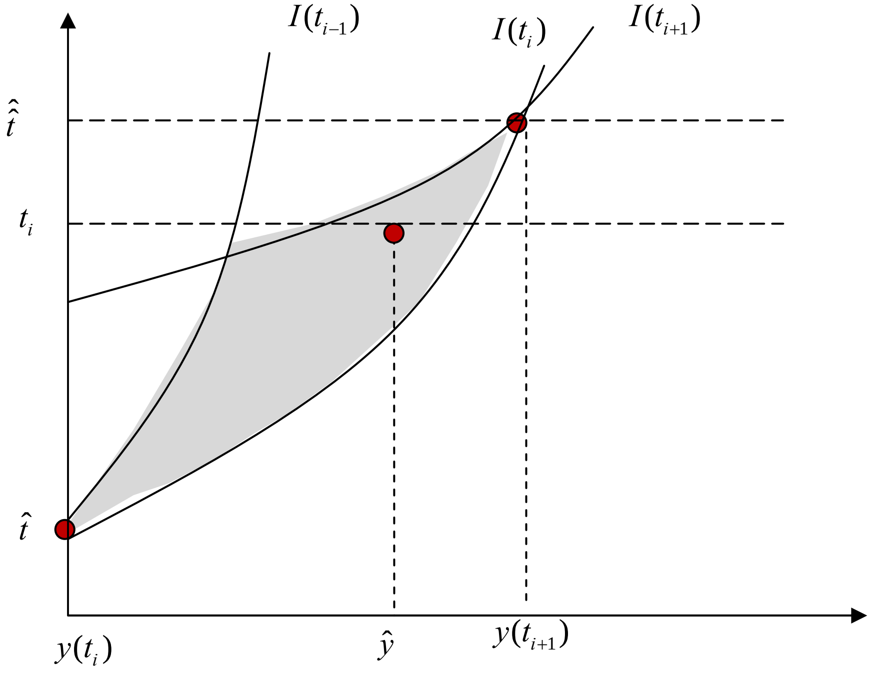

Fig 5.1: Applying the strong LCT

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[10] As Mailath et al have argued, the strong LCT is a controversial assumption. The idea is that if the type with the lowest signaling cost in a pool can credibly signal she will do so. But if this is the case, other types in the pool will worry about the response if they do not mimic the lowest cost type since the seller will know that they are not the high value type. But then the original argument is no longer so credible.

We next introduce an outside opportunity. Let $U^R(t)$ be the reservation utility of type *t*. We assume that it is efficient to choose the outside opportunity if and only if $t \le t^*$. Define $a(t_1, t_2) = E\{t \mid t \in [t_1, t_2]\}$ and $U^P(t \mid t^*) = a(t^*, t)$.

Suppose that $U^P(t \mid t^*) < U^R(t)$ for all $t > t^*$. This is depicted below. Suppose first that all types above t^* are separated. Then arguing as in the previous section, the slope of $U^S(t \mid k(t^*))$ is zero at t^* thus there are an interval of types better off pooling. It follows that there is some $k^{**} > k(t^*)$ and $\tau^{**} > t^*$ such that (i) $U^S(\tau^{**} \mid k^{**}) = U^P(\tau^{**})$, (ii) $U^S(t \mid k^{**}) > U^P(t \mid t^*),\ t \in [\tau^{**}, t_\infty]$.

Suppose that $\tau^{**} < t_\infty$ and consider a Nash Equilibrium in which all those above τ^{**} separate. By hypothesis $U^R(t) > U^P(t \mid t^*)$ for all $t > t^*$. Thus all types on the interval $[t^*, \tau^{**})$ are better off staying out than pooling. But if there is no pooling of types above t^*, all lower types are better off accepting their outside offers as well since signaling is

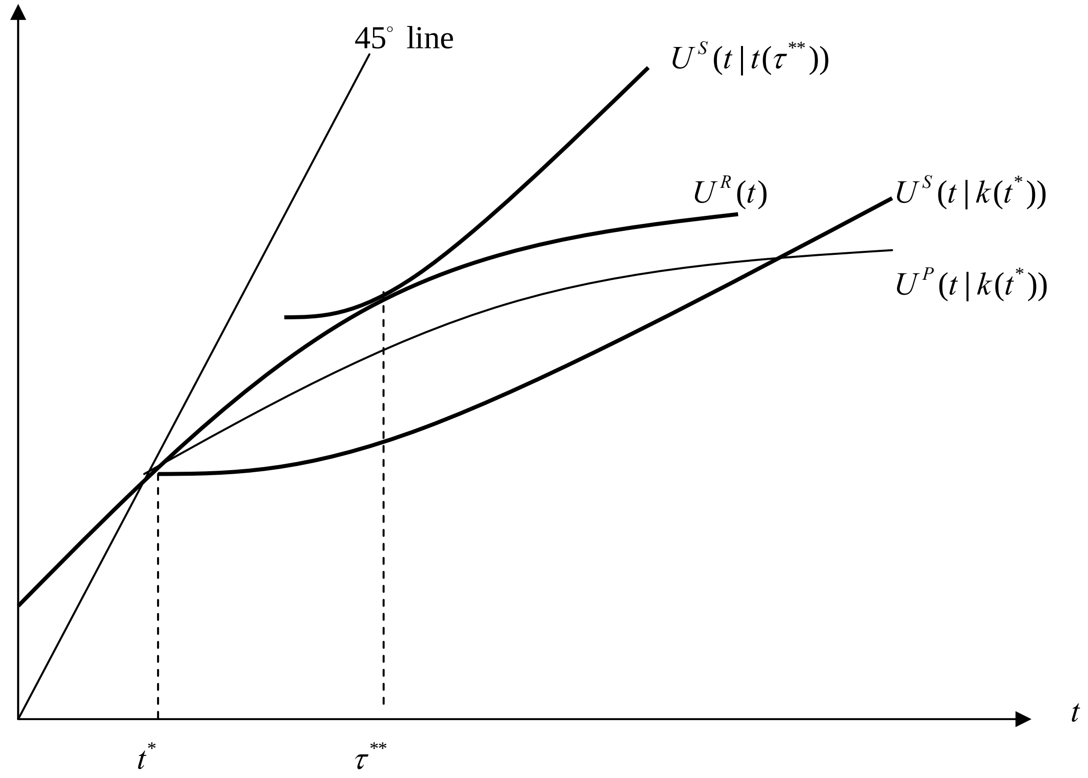

Fig. 5.2: Separating equilibrium

even more costly for them. Thus the Nash Equilibrium in which all those types above τ^{**} enter and are separated satisfies the LCT. Since there are no pools the equilibrium also satisfies the strong LCT.

The other alternative is that $\tau^{**} = t_\infty$ in which case there is no equilibrium with signaling. We thus have the following result.

Proposition 5.2: Existence of an equilibrium satisfying the strong LCT
Suppose that $t^* = U^R(t^*)$, $t > U^R(t),\ t > t^*$ and $U^P(t\,|\,t^*) < U^R(t)$ for all $t > t^*$. Then either no type signals or there exists a Nash equilibrium satisfying the strong LCT in which all those types who enter are separated.

6. Many finite types and a continuum of types

We begin by examining the conditions under which the LCT holds when the difference between types gets small.. We then consider a continuum of types. We make the following standard assumptions. The sender's payoff $u(t,\hat{t},y) \equiv U(t,r(\hat{t}),y)$ is third order differentiable in all its elements and is increasing in its first two arguments. Furthermore, the sender's payoff function under full information $u(t,\hat{t},y)$ is strict quasi-concave in y, so that there is a unique optimal signal y_i^* for each type t_i if the sender's type is known to the receiver. To rule out trivial cases, we assume that $u(t_i,t_{i+1,}y_{i+1}^*) > u(t_i,t_{i,}y_i^*)\ \forall i$. If this condition does not hold for every type, then it is a separating equilibrium in which each type chooses her complete information optimal signal. This condition must be satisfied if the types are sufficiently close.[11] The last standard assumption we make is the single crossing condition:

$$\frac{\partial}{\partial t}\frac{d\hat{t}}{dy}\bigg|_U = -\frac{\partial}{\partial t}\frac{u_3}{u_2} = -\frac{u_{13}u_2 - u_{12}u_3}{u_2^{\,2}} < 0$$

---

[11] Let $\Delta t = t_{i+1} - t_i \to 0$, then, since $u_3(t,t),y^*(t)) = 0$, $[u(t_i,t_{i+1},y_{i+1}^*) - u(t_i,t_i,y_i^*)]\big/\Delta t \to U_1 + U_2 > 0$.

The slope of the indifference curve through any pair $(y,\hat{t})$ is $\left.\frac{d\hat{t}}{dy}\right|_U = -\frac{u_3}{u_2}$ $\left.\frac{d\hat{t}}{dy}\right|_U = -\frac{u_3}{u_2}$.

The single crossing condition requires that this should decrease with type.

For simplicity, we make the following technical assumptions:

B1: u_{12} =0 for all $(t,\hat{t},y)$,

B2: u_{22} =0 for all $(t,\hat{t},y)$.

B1 and B2 are satisfied by many signaling models. They are not crucial for our main results to hold. Assumption (B1) and the single crossing condition imply that $u_{13} > 0$.

We now seek general conditions under which, if types t_i and t_{i+1} are separated, the LCT is satisfied

The equilibrium signals for t_i and t_{i+1} are depicted in Figure 6.1 below.

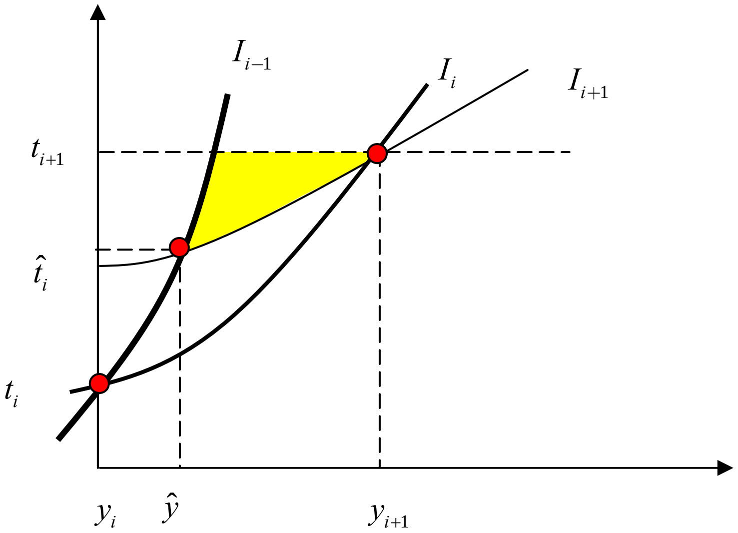

Fig. 6.1: Applying the LCT

Define $(\hat{y},\tilde{t})$ to be the intersection point of indifference curves I_{i-1} and I_{i+1} depicted in Figure 6.1. Then

$$\begin{cases} u(t_{i-1},\hat{t}_i,\hat{y}) = u(t_{i-1},t_t,y_i) \\ u(t_{i+1},\hat{t}_i,\hat{y}) = u(t_{i+1},t_{i+1},y_{i+1}) \\ u(t_i,t_i,y_i) = u(t_i,t_{i+1},y_{i+1}) \end{cases} \tag{6.1}$$

For all i, let $\overline{t}_i$ be the expected value of t_i and t_{i+1}, that is,

$$\overline{t}_i = \frac{[F(t_i)-F(t_{i-1})]t_i + F(t_{i+1}) - F(t_i)]t_{i+1}}{F(t_{i+1})-F(t_{i-1})} \tag{6.2}$$

Proposition 6.1: The tight separating equilibrium satisfies the LCT test if and only if for all types that separate , $\hat{t}_i > \overline{t}_i$. **:**

Proof: (Sketch) Consider the out-of-equilibrium signal $\hat{y} \in (y_i, y_{i+1})$. Then $S_i = \{t_i\}$ and $S_{i+1} = \{t_{i+1}\}$ and so $S_i \cup S_{i+1} = \{t_i, t_{i+1}\}$. For an offer to be acceptable by both types but not by type t_{i-1}, it must lie in the shaded region. Thus smallest acceptable point estimate of the types making the out-of-equilibrium signal is $\hat{t}_i$. If this exceeds $\overline{t}_i$ then LCT is satisfied. There is no local pool that can profitably deviate.

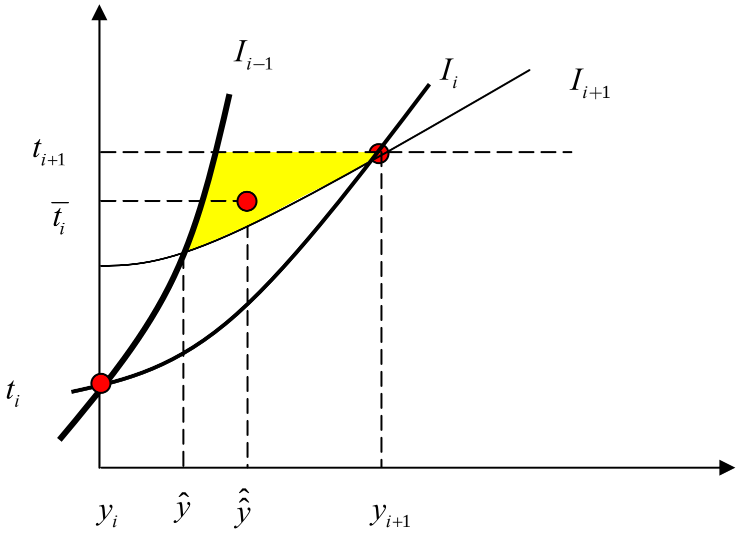

Fig 6.1b: Applying the LCT

Next suppose that $\hat{t}_i < \overline{t}_i$ we can choose $\hat{\hat{y}} > \hat{y}$ such that types t_i and t_{i+1} are both better off if offered $(\hat{\hat{y}}, \overline{t}_i)$ and all lower types are worse off. Moreover, given single crossing, if

$\hat{\hat{y}}$ is sufficiently close to the indifference curve for type t_{i+1} it will be below the indifference curve for all higher types. Thus the separating equilibrium fails the LCT.

QED

We now consider the limiting case of many types. To make the connection to the continuous type case where the type space is $T = [t_0, t_\infty]$ and the cumulative probability distribution function is $F(t)$ with $F'(t) > 0$ for all $t \in T$, we consider the following finite type version of the signaling model. Suppose there are N types ($N \geq 2$) where $t_1 = t_0$, $t_{i+1} = t_i + \delta$, $t_N = t_\infty$ and $\delta = \frac{t_\infty - t_0}{N-1}$. For this finite type case, given any δ let $F(s_i) = \Pr\{s \leq s_i\}$ so that when $\delta \to 0$ the cumulative probability distribution approximates the cumulative probability distribution in the continuous type case. We let $N = 2^k - 1$ for $k \geq 2$, that is, as k increases by one, each interval is divided into two even ones. We are interested in the limit case when $k \to \infty$, or $\delta \to 0$, as an approximation of the continuous type case.

A tight separating equilibrium $y_i = y(t_i)$ satisfies $u(t_i, t_i, y_i) = u(t_i, t_{i+1}, y_{i+1})$ for every t_i. Fix any t_i and let $\delta \to 0$, it must be that $(t_{i+1} - t_i)u_2 + (y_{i+1} - y_i)u_3 \to 0$. Thus, the equilibrium signaling schedule $y_i = y(t_i)$ satisfies $y'(t) = -u_2(t,t,y)/u_3(t,t,y)$.

To apply Proposition 6.1, for any k and $n \in \{\tilde{i}, \tilde{i}+1, \ldots 2^k - 1\}$, fix $t_n = t$ .. As k increases, the nearest types to *t*, $t_{n-1} = t - \delta(k)$ and $t_{n+1} = t + \delta(k)$, both get closer to *t*. Let $\bar{t}(t,\delta) = \bar{t}_n = E\{t \mid t = t_n \text{ or } t = t_{n+1}\}$ as defined in (6.2), and $\hat{t}(t,\delta) = \hat{t}_n$ be the solution to (6.1).

Lemma 6.2: *When* $\delta \to 0$, *(i)* $\bar{t}(t,\delta) \to t$ *; (ii)* $\bar{t}_2(t,\delta) \to 1/2$ *; (iii)* $\bar{t}_{22}(t,\delta) \to \frac{F''(t)}{2F'(t)}$.

Proof: See the Appendix.

Lemma 6.3: Suppose conditions **B1** and **B2** hold. When $\delta \to 0$, (i) $\hat{t}(t,\delta) \to t$; (ii) $\hat{t}_2(t,\delta) \to 1/2$; *(iii)* $\hat{t}_{22}(s,\delta) \to -\frac{u_2 u_{13} u_{33} + u_2 u_3 u_{133} + 2u_3 u_{13} u_{23} - 2u_3^2 u_{113} + 4u_3 u_{13}^2}{4u_3^2 u_{13}}$.

Proof: See the Appendix.

We have our main characterization result for the limiting finite type case.

Proposition 6.4: Suppose assumptions **B1** and **B2** hold. When $\delta \to 0$, a tight separating equilibrium satisfies the LCT at t if and only if

$$-\frac{u_2 u_{13} u_{33} + u_2 u_3 u_{133} + 2u_3 u_{13} u_{23} - 2u_3^2 u_{113} + 4u_3 u_{13}^2}{4u_3^2 u_{13}} > \frac{F''(t)}{F'(t)}. \tag{6.3}$$

Proof: See the Appendix.

The idea for Proposition 6.4 is as follows. Consider the consulting example discussed in Section 2, where we showed that one reason for non-existence is that the type distribution is tilted upwards too much. Nonexistence also arises if the indifference curves differ too little across types. Since the marginal rate of substitution between signal y and perception $\hat{t}$ is similar for the different types in that example, the indifference maps are similar and so indifference curves are close together. As a result, both types are better off deviating to $\hat{y}$ if the receiver believes that both may be choosing to deviate, thus violating the requirement of no credible deviation. However, if the marginal rate of substitution declines sufficiently rapidly with type, the indifference curve I_{i+1} in Figure 6.1 will be flat relative to the indifference curve I_i and I_{i-1}. Then the intercept of I_{i+1} and I_{i-1}, $\hat{t}_i$, will be above $\overline{t}_i$, the average type. In this case there is no deviation by types t_i and t_{i+1} such that they will be better off under the perception of $\hat{t}_i$ but not any other types.

Intuitively, the rate at which the marginal rate of substitution declines with *t* is a measure of signaling effectiveness. Thus Proposition 6.4 suggests that when signaling effectiveness is sufficiently large, the separating equilibrium will survive the LCT. This intuition is reflected in condition (6.3). Note that the slope of the indifference map is given by $MRS = -U_3/U_2$, and by Assumption **B1,** $\frac{\partial}{\partial s}MRS = \frac{-U_{13}}{U_2}$. The last term of the LHS of (6.3) (over the denominator), $-\frac{2U_{13}}{U_3} = -\frac{2U_{13}/U_2}{U_3/U_2}$, has exactly the same interpretation: a measure of how rapidly the MRS declines with *t*. From Figure 3.3, the critical value of $\tilde{v}_i$ depends on how rapidly the curve I_{i-1} increases with z and how slowly the curve I_{i+1} increases with z . Note that $\frac{\partial}{\partial v}MRS(s,v,z(s))\Big|_{v=s} = \frac{-U_{23}}{U_2}$ by Assumption **B2** and $\frac{\partial}{\partial z}MRS(s,s,z) = \frac{-U_{33}U_2 + U_3U_{23}}{(U_2)^2}$. The first and third terms of the LHS of (6.3) can be rewritten as

$$-\frac{U_{13}(U_3U_{23}-U_2U_{33})+U_3U_{13}U_{23}}{2U_3^2U_{13}} = -\frac{U_3U_{23}-U_2U_{33}}{2U_3^2} - \frac{U_{23}}{2U_3} = -\frac{\partial MRS/\partial z}{MRS^2} + \frac{\partial MRS/\partial v}{MRS}$$

Figuratively, when the curve I_{i-1} is more straight-up (large $\partial MRS/\partial v$) and the curve I_{i+1} is more flat (small $\partial MRS/\partial z$), there will be no credible deviation with the perception at $\bar{v}$. The RHS of (6.3) is the concavity of the distribution function of s , $G(s)$, normalized by its density function. Intuitively, the more concave $G(s)$ is (i.e., the smaller G'' is), the more probability mass on smaller s in any set of types, thus the smaller the expected value of any set of types. Consequently, the smaller G'' is, the less likely a deviation is credible.

We now turn to the continuum of agents. The type space is $S = [\underline{s}, \overline{s}]$ and the cumulative probability distribution function is $G(s)$ with $G'(s) > 0$ for all s . We will show that the condition for a signaling equilibrium to satisfy LCT at *t* will be exactly the same as those derived in the limiting finite type case studied in the preceding section.

The standard result in the literature (Riley, 1979; Mailath, 1987) shows that a separating equilibrium satisfies the following differential equation for active senders who choose to signal:

$$z'(s) = -\frac{U_2(s,s,z)}{U_3(s,s,z)} \tag{6.4}$$

A conceptual issue arises when equilibrium refinements are considered for continuous types. In the continuous type case, any signal $y \in [\tilde{z}, \overline{z}]$ is "on-equilibrium," which leaves no room for considering out of equilibrium beliefs in the conventional approaches to equilibrium refinements. However, thinking of the continuous type case as the limit of the finite type case with many very close types, it is easy to understand how an "on-equilibrium" signal can be alternatively interpreted as a deviating signal and how to check credibility of such deviations. Consider any "on-equilibrium" signal $\hat{y}$, and suppose the type of sender for this signal in the separating equilibrium is s_0. Suppose in a finite type version of the model, $s_n < s_0 < s_{n+1}$ where s_n and s_{n+1} are two consecutive types. Suppose the nearby types $s \in S_0 = \{s_n, s_{n+1}\}$ deviate to this signal, and this is correctly perceived by the receiver and so the perception of the average type is $\hat{s} = E[s \mid s \in S_0]$. The LCT requires that if given the perception $\hat{s}$, all the deviating types in S_0 can gain relative to their equilibrium payoffs while all other types cannot, then those nearby types can credibly deviate to $\hat{y}$. If a separating equilibrium does not allow any such credible deviations, then it satisfies the LCT. Another way of thinking about on-equilibrium deviations is as follows. If for an on-equilibrium signal there is such a deviation-perception pair $(\hat{y}, \hat{s})$ as described above, then those nearby types can credibly deviate to $\hat{y}$ by throwing away a small amount of money. Since other types will not gain by mimicking, throwing away money by types in S_0 can convey to the receiver that they are the types deviating to $\hat{y}$.

Formally, the LCT test in the continuous type case can be stated as follows

Local Credibility Test:

Consider a separating equilibrium schedule $z(s):[\tilde{s}, \overline{s}] \to [\tilde{z}, \overline{z}]$. For any signal $\hat{y} \in [\tilde{z}, \overline{z}]$, let $z(s_0) = \hat{y}$ and consider a small neighborhood of s_0, $S_0 \subset S$. Let $\hat{s} = E[s \mid s \in S_0]$. If

(*i*) $U(s, \hat{s}, \hat{y}) > U(s, s, z(s))$, for all $s \in \operatorname{int} S_o$, and

$(ii)\ U(s,\hat{s},\hat{y}) < U(s,s,z(s))$, for all $s \notin S_o$.

Then the signal-perception $(\hat{y},\hat{s})$ is credible. If the separating equilibrium $z(s)$ survives the LCT, then there cannot exist any credible signal-perception pair.

Since the continuous type case is viewed as an approximation of the case of many close finite types, we only need to check whether there are any credible deviations for very small intervals, in the sense that will be made precise below.

For any two types s and s' where $\tilde{s} \le s < s'$, suppose those in the interval $[s,s']$ pool at a certain signal y. Let $\overline{v}(s,s')$ be the expected type of this pool:

$\overline{v}(s,s') = \dfrac{\int_s^{s'} x dG(x)}{G(s') - G(s)}$. Let $v(s,s') \in [s,s']$ and $y(s,s') \in Y$ be a solution to

$$\begin{cases} U(s',v,y) = U(s',s',z(s')) \\ U(s,v,y) = U(s,s,z(s)) \end{cases} \tag{6.5}$$

The point $(y(s,s'),v(s,s'))$ is depicted below in Figure 6.1

Given this signal-perception pair, all those types in $[s,s']$ prefer the pool to their separating equilibrium payoff.

The types in $[s,s']$ cannot find a credible deviation if and only if the signal-perception pair of $(y(s,s'),\overline{v}(s,s'))$ is not credible. Therefore, the separating equilibrium

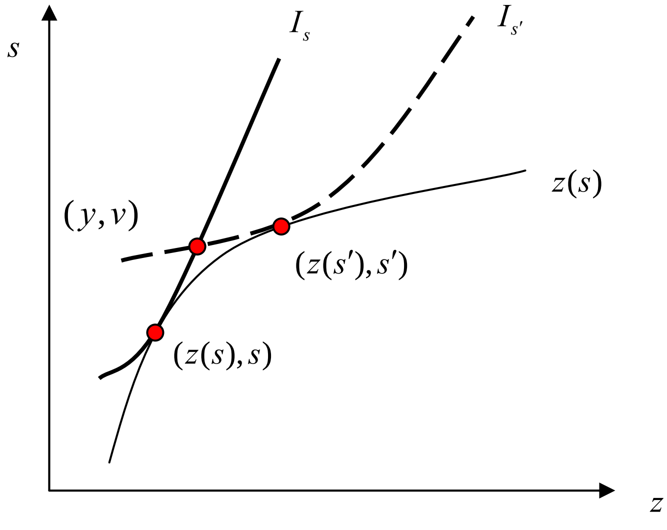

Fig. 6.2: Pool of types in $[s,s']$

$z(s)$ satisfies LCT if for any $s \in [\tilde{s}, \overline{s})$ and $s' > s$, $\overline{v}(s, s') < v(s, s')$ as $s' \to s$. Note that for any $s \in [\tilde{s}, \overline{s})$, $\overline{v}(s, s) = v(s, s) = s$. Furthermore, we have

Lemma 6.5: For any $s \in [\tilde{s}, \overline{s})$, (i) $\overline{v}_2(s, s) = 1/2$; (ii) $\overline{v}_{22}(s, s) = \frac{1}{6} \frac{G''(s)}{G'(s)}$.

Proof: See the Appendix.

Lemma 6.6: Suppose conditions **B1** and **B2** hold. (i) $v_2(s, s) = 1/2$; (ii)

$$v_{22}(s, s) = -\frac{-U_2 U_{13} U_{33} + U_2 U_3 U_{133} + 2U_3 U_{13} U_{23} - 2U_3^2 U_{113} + 4U_3 U_{13}^2}{12 U_3^2 U_{13}}$$

Proof: See the Appendix.

Proposition 6.7 below shows that the characterization result of Proposition 6.2 applies equally well to the continuous type.

Proposition 6.7: Suppose assumptions **B1** and **B2** hold. A separating equilibrium satisfies the LCT if and only if all the conditions of Proposition 6.4 hold.

Proof: See the Appendix.

Therefore, Proposition 6.4 and 6.7 show that the LCT can be applied to the continuous type model exactly as in the finite type model. While the continuous type model is easier to work with analytically in terms of characterizing the separating equilibrium, it should be viewed as an approximation of the situation with many close finite types. Our position is that it should be subject to the same scrutiny of credibility as finite type models, even though signals are literally "on-equilibrium" in the continuous type model. That both Propositions 6.4 and 6.7 share the same conditions demonstrates that this approach is valid.

Below we use three examples to illustrate how to apply our results to signaling models.

Example 1: The Spence education signaling model

A worker knows her own personal skill level or productivity, denoted by s. The labor market knows that s is drawn from distribution $G(s)$ on $[0,1]$. Her expected payoff is $U(s,\hat{s},y)=\hat{s}-C(s,y)$, where s is her productivity unknown to firms, $\hat{s}$ is her productivity perceived by firms and hence is also the wage offered to her by competing firms, and $y\in[0,\bar{y}]$ is the education signal the worker can choose. It is typically assumed that for all (s,y), (i) $C_1(s,y)<0$; (ii) $C_2(s,y)>0$; and (iii) $C_{12}(s,y)<0$. It can be verified that the single crossing and conditions B1 and B2 are satisfied.

To further simplify things, suppose $C(s,y)=ys^{-a}$, $a>0$. The marginal cost of signaling is $MC(s)=s^{-a}$ so the parameter a is the elasticity of the marginal cost of signaling with respect to type. Note that under complete information, workers of all types choose the minimal signal $\underline{y}=0$. Suppose the worker's reservation payoff is $U^R=\alpha+\beta s$, where $\alpha,\beta>0$. If $\alpha+\beta\geq 1$, no type can be better off in a separating equilibrium than if she accepts her outside alternative. Thus we assume that $\alpha+\beta<1$. Note that the lowest type in this case will not signal since $U^R(\underline{s})=\alpha>0=U(\underline{s},\underline{s},\underline{y})$. Moreover, with complete information a worker chooses not to signal if and only if $\alpha+\beta s<s$, that is, $s<s^*=\alpha/(1-\beta)$.[12]

To determine the minimum signaling type $\tilde{s}$ and the corresponding signal $\tilde{y}$, by part (ii) of Theorem 2 we have $\beta=U_1=ays^{-a-1}$ and $\alpha+\beta s=s-ys^{-a}$. Thus, $\tilde{s}=\alpha/(1-(\frac{a+1}{a})\beta)$. For $\tilde{s}<1$, we need to assume that $\alpha+\frac{1+a}{a}\beta<1$, That is, the elasticity of the signaling cost with respect to type (the parameter a) must be sufficiently large. Note that $\tilde{s}>s^*$, i.e., fewer types participate in the signaling market than under complete information. Moreover, the higher the elasticity of the marginal cost of signaling is (the larger a is), the lower is the minimum type who signals.

By part (iii) of Theorem 2, a separating equilibrium satisfies

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[12] With complete information a worker will choose $y^*=0$ should she decide to signal.

$$z'(s) = -\frac{U_2}{U_3} = \frac{1}{C_2} = s^a$$

Thus the signaling schedule is given by $z(s) = \frac{s^{a+1}}{a+1} + (\frac{\beta}{a} - \frac{1}{a+1})\tilde{s}^{a+1}$. Substituting for $z(s)$, it can be verified that for $s \in (\tilde{s},1]$, $U_1(s,s,z(s)) = az(s)/s^{1+a}$ is increasing in s, thus $U_1(s,s,z(s)) > \beta = dU^R/ds$. Hence the participation constraint is satisfied for all $s \in (\tilde{s},1]$.

Since $C(s,y) = ys^{-a}$, it can be checked that condition (6.3) can be simplified to

$$\frac{a-1}{s} > \frac{G''(s)}{G'(s)}$$

Therefore, by Theorem 2, the LCT will be satisfied if the signal effectiveness measured by a is sufficiently large and the type distribution is not tilted upward too much. When s is uniformly distributed, the right hand side is zero. Then when $a > 1$, the tight separating equilibrium satisfies the LCT.

Example 2: The reserve price signaling model

Cai, Riley and Ye (2007) study reserve price signaling in a fairly general auction environment allowing bidders' signals to be affiliated. A simpler version of the model is as follows. A seller of an indivisible good has private information about certain characteristics of the good that potential bidders do not know. Let $\theta \in \Theta \subseteq R^n$ be the seller's private information. The seller's own valuation of the good is $\gamma s(\theta)$, and the common value component of the bidders' valuations is $t(\theta) = s(\theta)$, where $\gamma > 0$. We normalize the range of $s(\theta)$ so that $s \in [0,1]$. Ex ante, the distribution of θ induces a distribution, $G(\cdot)$, for s.

Bidder i's valuation is $t + x_i$, where $x_i \in [0, \bar{x}]$ is the private value component that is known to himself only. The bidders' private signals $\{x_i\}$ are i.i.d. random variables with a distribution function $F(\bullet)$ and an everywhere positive density function $f(\bullet)$. Suppose the seller uses a sealed-bid second-price auction to sell the good; and she sets a reserve price r which determines m, the minimum type of bidder who enters the auction. Let $\hat{s}$ be the perceived type of the seller, i.e., the perceived common value component in bidders' valuations, then $m = r - \hat{s}$. Since the reserve price schedule can be recovered from the minimum type schedule through $r(s) = m(s) + s$, Cai, Riley and Ye (2004) focus on the minimum type schedule $m(\cdot)$ to characterize the signaling equilibrium. First, given the signal m and the perceived common value $\hat{s}$, the seller's expected payoff can be expressed as

$$U(s,\hat{s},m) = \gamma s F_{(1)}(m) + \hat{s}(1 - F_{(1)}(m)) + B(m) \quad (6.6)$$

where $F_{(1)}(\bullet)$ is the distribution function of the first order statistics, and $B(m) = m(F_{(2)}(m) - F_{(1)}(m)) + \int_m^{\bar{x}} x dF_{(2)}(x)$ and $F_{(2)}(\bullet)$ is the distribution function of the second order statistics. Thus the model fits into the standard signaling framework. It can be verified that the single crossing and **B1** and **B2** conditions are all satisfied.

If the seller does not sell the item in the auction, she may sell the item by some other means such as posted price or bargaining. Suppose the payoff implied by the best outside option is given by $U^R(s) = \alpha + \beta s$, where $\alpha, \beta > 0$, and $\beta < \gamma$. To determine the minimum signaling type $\tilde{s}$ and the corresponding signal $\tilde{m}$, by part (ii) of Theorem 2 we have $\beta = \gamma F_{(1)}(m)$ and $\alpha + \beta s = \gamma s F_{(1)}(m) + s(1 - F_{(1)}(m)) + B(m)$. Solving this equation system we have $\tilde{m} = F_{(1)}^{-1}(\beta / \gamma)$ and $\tilde{s} = \left(1 - \beta/\gamma\right)^{-1}\left(\alpha - B(\tilde{m})\right)$. For $\tilde{s} \in (0,1)$, we need to assume that $\left(1 - \beta/\gamma\right)^{-1}\left(\alpha - B(\tilde{m})\right) \in (0,1)$.

From Proposition 6.4, a separating equilibrium satisfies

$$m'(s) = -\frac{U_2(s,s,m(s))}{U_3(s,s,m(s))} = \frac{1 - F_{(1)}(m)}{(J(m) - (\gamma - 1)s) f_{(1)}(m)}$$

where $J(m) = m - \left(1 - F(m)\right)/f(m)$. From (6.6), it can be verified that condition (4.3) in this reserve price signaling model amounts to

$$m'(s)\left[\frac{J'(m)}{2\left((\gamma-1)s - J(m)\right)} + (2\gamma - 1)\frac{f_{(1)}(m)}{1 - F_{(1)}(m)}\right] > \frac{G''(s)}{G'(s)}. \quad (6.7)$$

In the special case where there are two bidders, each bidder's private value signal is distributed uniformly on [0,1], and $\gamma \in (1,2)$,[13] it can be verified that the solution to (4.1) and (4.2) is given by $\tilde{m} = \sqrt{\beta/\gamma}$ and $\tilde{s} = (1-\beta/\gamma)^{-1}\left[\alpha - \frac{1}{3} - \beta/\gamma + \frac{4}{3}(\beta/\gamma)^{3/2}\right]$. In this case, for $\tilde{s} < 1$, it requires that $(\beta/\gamma)^{3/2} < 1 - 3\alpha/4$. Thus, if either α or β/γ is too large, there is no signaling by any type, which is intuitive as then the outside option would be too attractive. Since $F(x)$ is uniform on [0, 1], by substituting $F_{(1)}(m) = m^2$, $f_{(1)}(m) = 2m$, and $J(m) = 2m-1$ into (6.7), we have

$$m'(s)\left[-\frac{1}{2m-1-(\gamma-1)s} + (2\gamma-1)\frac{2m}{1-m^2}\right] > \frac{G''(s)}{G'(s)}$$

If $G(\cdot)$ is concave, then the above inequality holds if the LHS is strictly positive. It can be verified that for a fixed β, this holds for relatively small γ within the relevant range (such that $\tilde{s} < 1$). Thus, else being equal, the smaller γ, the more likely that the signaling equilibrium can satisfy our LCT (when $G(\cdot)$ is concave).

In the two examples above, both of the regularity conditions **B1** and **B2** hold so we can directly apply Theorem 2 (iv). Below we illustrate via an example that our approach is still applicable even when those regularity conditions fail.

Example 3: An advertising signaling model

This example is adopted from Milgrom and Roberts (1986). A monopolistic firm can produce a good with a constant marginal cost c. It sells to a unit mass of consumers. The firm knows its product quality, denoted by s. Among the consumers, $a < 1$ are informed about s. The rest $1-a$ of consumers are uninformed of s, and their belief is given by the distribution $G(s)$ on $[\underline{s}, \overline{s}]$. For products of quality s, the consumers' inverse demand function is $p = s - bq$, where b is a positive parameter and q is the

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[13] Restricting γ within $(1,2)$ ensures that the equilibrium signaling schedule would not be truncated (see Theorem 3 in Cai, Riley, and Ye (2004)).

quantity. So, given price p, the demand of an informed consumer is $q_I = (s-p)/b$, and that of an uninformed consumer is $q_U = (\hat{s}-p)/b$, where $\hat{s}$ is his perception of s. The total demand is then $q = aq_I + (1-a)q_U$.

Suppose the firm spends z on advertising, which leads to a perception of $\hat{s}$ by the uninformed consumers. By choosing an optimal price given $\hat{s}$, the firm's maximum profit is $U = \frac{[as+(1-a)\hat{s}-c]^2}{4b}$. Consider a possible separating signaling schedule $z(s)$. The firm's payoff function is

$$U(s,\hat{s},z) = \frac{[as+(1-a)\hat{s}-c]^2}{4b} - z$$

Suppose the firm's reservation payoff is $U^R = \alpha + \beta s$, where $\alpha, \beta > 0$. When $\alpha + \beta \underline{s} > (\underline{s}-c)^2/(4b)$, the lowest type will not signal. The minimal signaling type in equilibrium can be solved by appealing to the conditions of Proposition 5.2, giving $\beta = U_1(s,s,z(s)) = a(s-c)/(2b)$, or $\tilde{s} = c + 2b\beta/a$. The associated minimum signal is $\tilde{z} = (\tilde{s}-c)^2/(4b) - \alpha - \beta\tilde{s}$.

It can be checked that the single crossing condition holds. From the FOC for a separating equilibrium,

$$z'(s) = -\frac{U_2(s,s,z(s))}{U_3(s,s,z(s))} = \frac{(1-a)(s-c)}{2b}$$

Since $U_1(s,s,z(s))$ is increasing in s, thus $U_1(s,s,z(s)) > \beta = dU^R/ds$ for $s \in (\tilde{s},1]$. Hence the participation constraint is satisfied for all types above $\tilde{s}$.

From the firm's payoff function, we have $U_{12} = a(1-a)/2b$ and $U_{22} = (1-a)^2/2b$. Since this model does not satisfy conditions **B1** and **B2**, we cannot apply Theorem 2 (iv) directly. However, using the same method that we follow in showing Theorem 2 (iv), it is not difficult to derive conditions under which the LCT is satisfied in this example. We summarize the conditions in the following proposition:

Proposition 3: *In the advertising signaling model, the separating equilibrium satisfies the LCT if* $G''(s) < 0$ *for* $s \in [\tilde{s}, \bar{s}]$, *where* $\tilde{s} = c + 2b\beta/a$ *is the minimal signaling type.*

Proof: See the Appendix.

Because of the quadratic function form of the firm’s payoff function and other special features of this model (e.g., many cross derivatives are zero), the measure of signal effectiveness is constant and equals zero. So whether the tight separating equilibrium satisfies the LCT only depends on the type distribution. A special case commonly studied in applications is when the type is uniformly distributed. Then $G'' = 0$ for $s > \underline{s}$. By our definition, however, the LCT is not satisfied.

7. Concluding Remarks

Except in the special case of perfect correlation between the sender’s true type and the value to the signal receiver, standard refinements (Intuitive Criterion, Divinity, Stability) are not applicable. We argue that to have any “bite” at all, a refinement is needed in which the signal receiver takes into account the way sender types are distributed and allows the possibility of deviations by a pool of sender types (in addition to single-type deviations). We then propose a Local Credibility Test which is somewhat stronger than the Cho and Kreps Intuitive Criterion but milder than the Grossman-Perry Criterion. Allowing deviations by a pool of “nearby” types, the LCT gives consistent solutions for any positive, though not necessarily perfect correlation between the signal sender’s signaling cost types and the receiver’s expected values. Besides this, the LCT has two additional advantages. It avoids selecting separating equilibria when they do not make sense, thus providing economically sensible answers to the equilibrium selection problem in signaling models. Moreover, it applies equally well in cases of finite and continuous types, making it applicable to many signaling applications that are formulated in continuous type models. In this paper we provide conditions under which a tight separating equilibrium satisfies the LCT. These conditions are the more likely to be met, (a) the less rapidly the density increases or the more rapidly the density decreases with type, and (b) the more rapidly the marginal cost of signaling decreases with type. We illustrate the applicability of the LCT in three examples of signaling models.

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Appendix (incomplete)

Lemma 4.6: If for some τ , $\frac{H'(\tau)}{H(\tau)} > \frac{F'(\tau)}{F(\tau)}$ and for all $\tau > \tau$, $\frac{H''(t)}{H'(t)} > \frac{F''(t)}{F'(t)}$, then

$\frac{H'(\tau)}{H(\tau)} > \frac{F'(\tau)}{F(\tau)}$ for all $t > \tau$.

Proof: Since $\frac{H'(\tau)}{H(\tau)} > \frac{F'(\tau)}{F(\tau)}$, either the inequality holds for all larger *t* or there is some $\hat{\tau}$

such that (i) $\frac{H'(\hat{\tau})}{H(\hat{\tau})} = \frac{F'(\hat{\tau})}{F(\hat{\tau})}$ and (ii) $\frac{d}{dt}(\frac{H'(t)}{H(t)}) \le \frac{d}{dt}(\frac{F'(t)}{F(t)})$ at $t = \hat{\tau}$.

$$\frac{d}{dt}(\frac{H'}{H}) = \frac{H''H - (H')^2}{H^2} = \frac{H'}{H}(\frac{H''}{H'} - \frac{H'}{H}) .$$

Thus at $t = \hat{\tau}$

$$\frac{d}{dt}(\frac{H'}{H}) = \frac{H''H - (H')^2}{H^2} = \frac{F'}{F}(\frac{H''}{H'} - \frac{F'}{F}), \text{ since } \frac{H'(\hat{\tau})}{H(\hat{\tau})} = \frac{F'(\hat{\tau})}{F(\hat{\tau})} .$$

By hypothesis $\frac{H''}{H'} > \frac{F''}{F'}$. Then at $t = \hat{\tau}$

$$\frac{d}{dt}(\frac{H'}{H}) > \frac{F'}{F}(\frac{F''}{F'} - \frac{F'}{F}) = \frac{d}{dt}(\frac{F'}{F}) .$$

But this contradicts (ii).

QED

Proposition 4.6: Weak and strong signals

Let $U_i(t),\ i = 1,2$ be equilibrium payoffs when the cost of signaling is

$C_i(t, y) = y / H_i(y)$ and types signal if and only if $t \ge \tau$. If $\frac{H_1'(t)}{H_1(t)} > \frac{H_2'(t)}{H_2(t)},\ t > \tau$ then

$U_i(t) > U_2(t),\ t > \tau$.

Proof: Define $e_i(t) = H_i'(t) / H_i(t)$. Then $e_1(t) > e_2(t),\ t > \tau$ and

$$\frac{dU_i}{dt} = e_i(t)(t - U_i(t) . \tag{8.1}$$

Since types on the interval $[t_0, \tau]$ do not signal, they have the pooling payoff $U^P(\tau)$. Then $U_1(\tau) = U_2(\tau)$. Since $e_1(\tau') > e_2(\tau')$, we have the following result.

$$U_1(\tau') = U_2(\tau'), \text{ for } \tau' > \tau \Rightarrow \frac{d}{dt}U_1(\tau') > \frac{d}{dt}U_2(\tau'). \qquad (8.2)$$

Suppose that there is some interval $(\tau, \tau']$ over which $U_1(t) < U_2(t)$. Then $t - U_1(t) > t - U_2(t)$ and since $e_1(t) > e_2(t)$ t follows from (8.1) that $\frac{d}{dt}U_1(t) > \frac{d}{dt}U_2(t)$ on $(\tau, \tau']$. But then $U_1(t) > U_2(t)$ on this interval, contradicting our initial hypothesis. Thus there exists some interval $(\tau, \tau']$over which $U_1(t) > U_2(t)$.

Either this inequality holds for all larger *t* or it holds over some interval (τ, τ'') and $U_1(\tau'') = U_2(\tau'')$. Then

$$\frac{d}{dt}U_1(\tau'') \leq \frac{d}{dt}U_2(\tau'').$$

But this contradicts (8.2) so $U_1(t) > U_2(t),\ t \in (\tau, t_\infty]$ after all.

QED